The Elastic Length: Key To The Analysis Of Multi-Layered Concrete Pavement Structures

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ABSTRACT: Westergaard introduced the term "elastic length" in his solution for the loaded slab on an elastic foundation. It will be shown that the equivalent elastic length "l_{eq}" of the structure is mandatory to solve multi-layered concrete pavements:

- For design purposes l_{eq} depends on the characteristics of the layers and the interface conditions between two adjacent layers.
- For the determination of the thermal stresses, l_{eq} further depends on the thermal gradient and the thermal expansion coefficients of the pavement materials.
- The automatic back-calculation of a deflection basin requires a seed l_{eq}.
- The translation of displacements measured in a slab with a hole to a full slab, followed by the determination of the stresses, again is based on l_{eq}.

KEY WORDS: Concrete, Multi-Layer, Design, Elastic Length, Stresses, Back-Calculation

1. THE MULTI-LAYERED PAVEMENT

Modern concrete pavements consist of several structural layers. A surface layer made of continuously reinforced concrete or jointed concrete slabs, an intermediate bituminous bonding layer and a lean concrete base layer.

The design of a multi-layered structure can be performed using the models for single slabs if the total thickness of the structure can be regarded as small in comparison with the global body. Mathematically this condition requires that the radius of curvature *($\rho = D/M$)* can be considered as large against the thickness of the structure, in other words, that the deflection at the surface of the upper layer can be considered to be equal with the deflection at the bottom of the lower layer.

1.1. The slab on a Pasternak foundation

The equilibrium equation of a slab subjected to a vertical load and resting on a Pasternak foundation is nowadays well known

$$\nabla^4 w - \frac{G}{D}\nabla^2 w + \frac{kw}{D} = \frac{p}{D} \tag{1}$$

where w is de deflection for any x,y, G is Pasternak's shear modulus and k Westergaard's subgrade modulus. In polar co-ordinates, the solution of (1) is

$$w = \frac{pa}{kl} \int_0^\infty \frac{J_0(mr/l) J_1(ma/l)}{m^4 + 2gm^2 + 1} dm \tag{2}$$

where p is the pressure of the load uniformly distributed over a circular area of radius a, l, the elastic length of the structure, is equal to $(D/k)^{1/4}$, D is the stiffness of the slab and $g = Gl^2/2D$. In the axis of the load (2) simplifies into

$$w = \frac{pa}{kl} \int_0^\infty \frac{J_1(ma/l)}{m^4 + 2gm^2 + 1} dm \tag{3}$$

and the moment writes

$$M_r = \frac{pal}{2}(1+\mu) \int_0^\infty \frac{J_1(ma/l)}{m^4 + 2gm^2 + 1} dm \tag{4}$$

1.2. The multi-layered structure on a Pasternak foundation.

The multi-layered structure can easily be deduced from the single slab model. One only needs to replace the elastic length of the slab by the elastic length of the multi-layered structure (Van Cauwelaert, 2001). As the mathematical derivation can be found elsewhere, we only present the results for a two-layered structure.

In the case of a two-layer with slip at the interface, the stiffness of the structure is the sum of the stiffness' of both layers:

$$D_{12slip} = D_1 + D_2 \tag{5}$$

$$l_{eq,slip} = \left(\frac{D_{12slip}}{k} \right)^{1/4} \tag{6}$$

In the case of a two-layer with friction at the interface

$$D_{12\,fric} = \frac{E_1 I_{12}}{1 - \mu_1^2} \tag{7}$$

where I_{12} is the moment of inertia of a T-section. The thickness of the vertical bar of the T-section is equal to the ratio of the moduli $E_2/E_1.(1-\mu_1^2)/(1-\mu_2^2)$.

$$l_{eq,\,fric} = \left(\frac{D_{12\,fric}}{k} \right)^{1/4} \tag{8}$$

In the case of a two-layer with partial friction (*F friction, (1 – F) slip)* at the interface

$$D_{12\,part} = (1-F) D_{12slip} + F D_{12\,fric} \tag{9}$$

$$l_{eq,part} = \left(\frac{D_{12\,part}}{k} \right)^{1/4} \tag{10}$$

Deflection and moment in the axis of the load are given by relations (3) and (4) wherein l is replaced by l_{eq}.

1.3. Equivalent structures.

The multi-layer model developed in paragraph 1.2 allows to compute stresses and deflections anywhere in the structure using one of the several existing computer programs. However, the parameter l_{eq} reveals more than that. It allows in a very simple way to determine equivalent structures.

If one has designed, for example, a two-layered structure consisting in a concrete slab on a lean concrete base with the assumption of full friction at its interface ($E_1 = 30000\ N/mm^2$, $E_2 = 15000\ N/mm^2$, $h_1 = 200\ mm$, $h_2 = 250\ mm$). What would be the thickness of the base (h_3) of an equivalent structure in the assumption of full slip at the interface?
For this simple problem with one unknown we shall state that the center deflections of both structures must be equal: $w_{slip} = w_{fric}$.
The required equations are

$$l_{eq,slip} = l_{eq,fric}$$

$$D_{12slip} = D_{12fric}$$

$$\frac{E_1 h_1^3}{12} + \frac{E_2 h_3^3}{12} = E_1 I_{12}$$

One finds in this particular case $h_3 = 475\ mm$.

In a second example consider an initial structure ($E_1 = 35000\ N/mm^2$, $E_2 = 15000\ N/mm^2$, $h_1 = 200\ mm$, $h_2 = 250\ mm$) with the full slip assumption for which we want to design an equivalent structure with other materials ($E_3 = 30000\ N/mm^2$, $E_4 = 10000\ N/mm^2$). What are the required thickness' (h_3, h_4) of the equivalent structure?
Here we need two relations: equality of the deflections and of equality of the bending stresses at the bottom of the concrete slab.
The required equations are

$$D_3 + D_4 = D_1 + D_2$$

$$\frac{6}{h_3^2}\frac{D_3}{D_3 + D_4} = \frac{6}{h_1^2}\frac{D_1}{D_1 + D_2}$$

The resulting thickness' are: $h_3 = 233\ mm$, $h_4 = 238\ mm$.

One could also postulate in previous example that the allowed bending strength of the slab concrete with $E_3 = 30000\ N/mm^2$ is only 90 % of the strength of the slab with $E_1 = 35000\ N/mm^2$. The required equations are:

$$D_3 + D_4 = D_1 + D_2$$

$$\frac{6}{h_3^2}\frac{D_3}{D_3 + D_4} = 0.90\frac{6}{h_1^2}\frac{D_1}{D_1 + D_2}$$

The resulting thickness' are: $h_3 = 210\ mm$, $h_4 = 287\ mm$.

2. THERMAL STRESSES IN A MULTI – LAYERED STRUCTURE

The theory of the determination of thermal stresses in a single or a multi-layered structure on a Pasternak foundation has been developed by Lemlin et al. (2003). We only recall here the minimum required by our purpose.

The determination of thermal stresses in a concrete slab is based on next differential equilibrium equation

$$\nabla^4 w - \frac{G}{D}\nabla^2 w + \frac{kw}{D} = 0 \qquad (11)$$

The solution of the equation depends on the value of the parameter $g = Gl^2/2D$. It contains 4 integration constants to be determined by the boundary constants.
The principal boundary equation expresses that in order to suppress the curvature of the slab due to thermal stresses moments have to be applied at the boundaries of the slab: on the boundaries of a slab of great length such moments that $M_c = D/R = D\alpha_0\tau$ (Timoshenko, 1951, § 1) and on the boundaries of circular or rectangular slabs such moments that $M_c = D/R(1+\mu) = D\alpha_0\tau\,(1+\mu)$ where D is the stiffness of the slab and μ its Poisson's ratio (Timoshenko, 1951, § 11).

The extension to a multi-slab system depends on the structural conditions of the multi-slab. Let us consider that the radii of curvature of the different layers are different (variable gradient with depth, different thermal dilatations), but that due to the weight of the slabs and of the applied loads the slabs remain in contact in the vertical direction. In that case one shall consider the system as a multi-layered structure with slip at the interfaces. Consider the two-layered structure represented on figure 1. The total moment acting on the border of the structure is given by next relation

$$M_{ctotal} = M_{c1} + M_{c2} \qquad M_{ctotal} = D_1\alpha_{01}\tau_1 + D_2\alpha_{02}\tau_2$$

The equilibrium equation is deduced from figure 1.

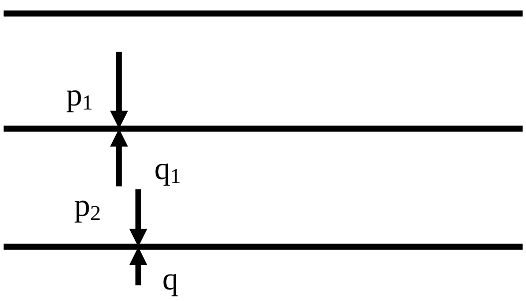

Figure 1: Equilibrium equation of a two-layer

The equilibrium equation of the upper slab is

$$\nabla^4 w_1 = -\frac{q_1}{D_1} \qquad (12)$$

That of the lower slab

$$\nabla^4 w_2 = \frac{p_1 - q}{D_2} \qquad (13)$$

The slabs remain vertically in contact. Hence

$$w_1 = w_2 = w \qquad (14)$$

Adding the two equilibrium equations yields

$$(D_1 + D_2)\nabla^4 w = -q_1 + p_1 - q = -q \qquad (15)$$

Let $D_{eq} = D_1 + D_2$ and $q = kw - G\nabla^2 w$.

The equilibrium equation of the equivalent structure writes

$$\nabla^4 w - \frac{G}{D_{eq}} \nabla^2 w + \frac{k}{D_{eq}} = 0 \qquad (16)$$

With $k/D_{eq} = 1/l_{eq}^4$. The multi-slab can further be resolved in the same way as a single slab Our conclusion will be very simple. How could we determine thermal stresses in a multi-slab without the tool of an equivalent elastic length?

3. BACKCALCULATION OF A CONCRETE STRUCTURE ON A PASTERNAK FOUNDATION.

3.1. Backcalculation considering the load as a point load.

Consider a slab of infinite extend. The deflection integral is given by relation (2).

$$w(r) = \frac{pa}{kl} \int_0^\infty \frac{J_0(mr/l) J_1(ma/l)}{m^4 + 2gm^2 + 1} dm \qquad (17)$$

If the load can be considered as a single load *P*, the relation simplifies in

$$w(r) = \frac{P}{2\pi k l^2} \int_0^\infty \frac{m J_0(mr/l)}{m^4 + 2gm^2 + 1} dm \qquad (18)$$

In r = 0, the integral can be analytically solved.
For example if $g < 1$

$$w(0) = \frac{P}{4\pi k l^2} \left[\frac{\pi}{2} - arctg \frac{g}{\sqrt{1-g^2}} \right] \frac{1}{\sqrt{1-g^2}} = \frac{P}{4\pi k l^2} w_o \qquad (19)$$

The value of *k* is then given by

$$k = \left[\frac{P}{4\pi \sqrt{D}} \frac{w_o}{w(0)} \right] \qquad (20)$$

Hence for any pair of *E* and *G* values, the value of *k* can be computed using relation (20). The backcalculation procedure is then reduced to the determination of 2 unknowns in stead of 3. The procedure could then be developed as follows. Given a seed value E_0, compute the *g* – value and automatically the *k – value* giving the best fit. Repeat for other *E* – values. However this method is inaccurate. The centre deflection of a concrete slab differs considerably according to the way the load is applied. Hence equation (19) cannot be utilised with a falling weight deflectometer. At a sufficient great distance of the load, the way the load is applied has no influence more on the deflection (principle of de St Venant). But there, the analytical solution of integral (18) is no more a simple relation in *k* so that the method loses its simplicity and hence cannot longer be utilised.

3.2. Backcalculation considering the load as a distributed load.

The deflection at any distance of the load is given by relation (2)

$$w(r) = \frac{pa}{kl} \int_0^\infty \frac{J_0(mr/l) J_1(ma/l)}{m^4 + 2gm^2 + 1} dm \qquad (21)$$

This integral can be solved analytically for $g > 1$.

If $r < a$

$$w(r) = \frac{pa}{2kl\sqrt{g^2 - 1}}\left(\frac{I_0(r\alpha / l)K_1(a\alpha / l)}{\alpha} - \frac{I_0(r\beta / l)K_1(a\beta / l)}{\beta}\right) \tag{22}$$

If $r > a$

$$w(r) = -\frac{pa}{2kl\sqrt{g^2 - 1}}\left(\frac{I_1(r\alpha / l)K_0(a\alpha / l)}{\alpha} - \frac{I_1(r\beta / l)K_0(a\beta / l)}{\beta}\right) \tag{23}$$

It is obvious that the value of k cannot directly be deduced from relations (22) or (23). However, the problem can be solved choosing as seed variable the elastic length and no longer the modulus. The procedure could then be developed as follows. Given a seed value l_0, compute the g – value, and automatically by (21) the *k – value*, giving the best fit. Repeat for other l – values until the difference between computed and measured deflections reaches an absolute minimum. E and G then can be calculated from the obtained values for l, g and k. The flow sheet, represented in appendix, indicates the sequence of the computations.

The strength of this procedure is that, while searching for the best fit, it overflows al possible values of the variables. It does not stop at a local minima, as some other algorithms sometimes do, but computes further until the minimum minimorum is reached. Further it reduces a problem if three variables to the more simple problem of two variables.

3.3. Backcalculation of a multi-layered structure.

It is obvious that when choosing the elastic length as seed variable for the backcalculation of the characteristics of a single slab, we can as well choose the equivalent elastic length for the backcalculation of the characteristics of a multi-slab.
The only difference is the look of equation (21), that here writes

$$w(r) = \frac{pa}{kl_{eq}}\int_0^\infty \frac{J_0(mr / l_{eq})J_1(ma / l_{eq})}{m^4 + 2gm^2 + 1}dm \tag{24}$$

4. THE OVALISATION TEST

The ovalisation test is a test procedure regularly utilised in France, however less known outside. Considering its interest, we shall summarise the method.

4.1. The ovalisation test (H. Goacolou et. al., 1983).

A cylindrical device with 3 horizontal electronic displacement gages (0°, 45°, 90°) is introduced in a cylindrical hole drilled in a pavement. The device can be placed at any depth. One measures the displacements under a moving truck. One computes the corresponding strains in the axis of a load on a full pavement, which allow deducing the mechanical characteristics of the pavement and the interface conditions.

4.2. Computation method (R. Kobisch, C. Peyronne, 1979).

The computation method is based on two models. Model I allows to take into account eventual disymmetries, Model II takes into account the influence of the subgrade on the stress distribution in the slab.

Model I consists of a plate with a circular hole submitted to a uniform tension of magnitude σ_L in the x – direction and of magnitude σ_T in the y – direction. Timoshenko's solutions for a hollow cylinder (1970, § 28) and for a plate with a circular hole (1970, § 35) allow to determine the relative variation of the diameter of the hole by

$$\varphi_\theta = \frac{\Delta\Phi_\theta}{\Phi} = \frac{\sigma_L}{E}\left[(1+\lambda)+2(1-\lambda)\cos 2\theta\right] \tag{25}$$

where $\lambda = \sigma_L/\sigma_T$.
Often σ_L coincides with the traffic direction and σ_T with the direction orthogonal to traffic.
If this is not the case, the principal stresses S_L and S_T can be deduced from (25) using Mohr's circle.

For $\theta = 0°$, $\varphi_L = \frac{\Delta\Phi_L}{\Phi} = \frac{\sigma_L}{E}[3-\lambda]$, $R_L = \frac{\varphi_L}{\frac{\sigma_L}{E}} = 3-\lambda$

For $\theta = 90°$, $\varphi_T = \frac{\Delta\Phi_T}{\Phi} = \frac{\sigma_L}{E}[3\lambda-1]$, $R_T = \frac{\varphi_T}{\frac{\sigma_T}{E}} = 3-\frac{1}{\lambda}$

Hence $\lambda = (3\varphi_T - \varphi_L)/(3\varphi_L - \varphi_T)$. If $\lambda = 1$, the case is symmetric and $R_S = R_L = R_T = 2$.
In the case of a full pavement, the strains can be obtained applying Hooke's law

$$\varepsilon_L = \frac{\sigma_L}{E}(1-\lambda\mu) \qquad \varepsilon_T = \frac{\sigma_L}{E}(\lambda-\mu) \tag{26}$$

We define

$$\Psi_L = \frac{\varphi_L}{\varepsilon_L} = \frac{3-\lambda}{1-\lambda\mu} \qquad \Psi_T = \frac{\varphi_T}{\varepsilon_T} = \frac{3\lambda-1}{\lambda-\mu} \tag{27}$$

If $\lambda = 1$, the case is symmetric and $\Psi_S = \Psi_L = \Psi_T = 2/(1-\mu)$.

Model II consists of a slab with a hole (diameter b) on a Winkler foundation (subgrade reaction modulus k) and subjected to a circular load (diameter a, pressure p).
The equilibrium equation writes in polar co-ordinates

$$\nabla^4 w + \frac{k}{D} w = \frac{p}{D} \tag{28}$$

The solution of (28) is

$$w = \frac{pa}{kl}\int_0^\infty \frac{J_0(mr/l)J_1(ma/l)}{m^4+1}dm + Af_1(w) + Bf_2(w) \tag{29}$$

$$f_1(w) = \frac{ber(r/l)}{2} + \frac{2}{\pi}\left\{bei(r/l)[ln(r/2l)+\gamma] - \frac{(r/2l)^2}{1!1!}\cdots\right\}$$

$$f_2(w) = \frac{bei(r/l)}{2} - \frac{2}{\pi}\left\{ber(r/l)[ln(r/2l)+\gamma] + \frac{(r/2l)^4}{2!2!}\cdots\right\}$$

the constants A and B are determined by the boundary conditions: for $r = b$, $M = T = 0$.

We have determined by Model I that in the symmetric case $R_S = \frac{\Delta\Phi / \Phi}{\sigma_L / E} = 2$, where σ_L is the uniform tension stress applied to a plate with a hole.

We shall now determine, using Model II, the value of the ratio $R = \frac{\Delta\Phi / \Phi}{\sigma_0 / E}$ where σ_0 is the bending stress in the axis of the load at the bottom of a full slab resting on a Winkler foundation.

A little algebra yields following results.

Since the case is symmetric $\frac{\Delta\Phi}{\Phi} = \frac{2u}{2r} = \varepsilon_\theta$. At the edge of the hole, $\varepsilon_\theta = \frac{\sigma_\theta}{E}$.

$$\frac{\sigma_\theta}{E} = \frac{v}{(1-\mu^2)} \frac{qa}{l^3 k} w_1(b,\mu,l) \tag{30}$$

where $q = pa^2/(a^2 - b^2)$

$$\frac{\sigma_0}{E} = \frac{v}{(1-\mu^2)} \frac{1+\mu}{2} \frac{pa}{l^3 k} w_2(a,l) \tag{31}$$

Hence

$$R = \frac{q}{p} \frac{1+\mu}{2} \frac{w_1(b,\mu,l)}{w_2(a,l)} \tag{32}$$

The value of R depends only of the parameters a, b, l and μ.

The dimensions a and b are known, Poisson's ratio μ can generally be estimated and the elastic length l can be backcalculated.

The strains in the axis of the load in a full plate can then be determined from the relative displacements measured in the bored hole by next relations

$$\varepsilon_{0L} = \frac{\Delta\Phi_L}{\Phi} \frac{1-\lambda\mu}{3-\lambda} \frac{2}{R} = \frac{\Delta\Phi_L}{\Phi} \cdot correctL \tag{33}$$

$$\varepsilon_{0T} = \frac{\Delta\Phi_T}{\Phi} \frac{\lambda-\mu}{3\lambda-1} \frac{2}{R} = \frac{\Delta\Phi_T}{\Phi} \cdot correctT \tag{34}$$

and in the symmetric case

$$\varepsilon_0 = \frac{\Delta\Phi}{\Phi} \frac{1-\mu}{2} \frac{2}{R} \tag{35}$$

The first term $\Delta\Phi/\Phi$ is the measured variation of hole diameter, the second term $(1-\mu)/2$ reduces the values of the strains measured at the edge of the hole to the values of the strain in the axis of a full plate, the third term $2/R$ corrects the value of the strain resulting from a stress distribution in a plate with a hole to the value of the strain resulting from the bending stress σ_0 at the bottom of a full slab on a Winkler foundation subjected to a distributed load

4.3. Conclusion.

It is remarkable that, besides the known geometrical data (load and bore hole diameters), only the knowledge of the value of the elastic length is required in order to determine the strains in a full slab based on the variations of the diameter of a bore hole observed in situ.

It is clear that in the case of a multi-slab the same method may be applied using the equivalent elastic length l_{eq}.

The method for determining the elastic length, based on deflection measurements, is developed in paragraph 3.

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NOTE

Computer programs illustrating all the presented theoretical developments are available on request at info@pavers.nl .

APPENDIX

Flow sheet of the
backcalculation of *E*, *k* and *G*
using the elastic length as seed variable